ON A CLASS OF GROUPS OF PRIME-POWER ORDER

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ABSTRACT

Let G be a finite p-group, $d(G)=\dim H^1(G, Z_p)$ and $r(G) = \dim H^2(G, Z_p)$. Then $d(G)$ is the minimal number of generators of G, and we say that G is a member of a class $\mathcal{G}_p$ of finite p-groups if G has a presentation with $d(G)$ generators and $r(G)$ relations. We show that if G is any finite p-group, then G is the direct factor of a member of $\mathcal{G}_p$ by a member of $\mathcal{G}_p$.

1.

Let G be a finite p-group. We have

$$d(G) = \dim H^1(G, Z_p),$$
$$r(G) = \dim H^2(G, Z_p),$$

$d(G)$ being the minimal number of generators of G. If there is a presentation

$$G = F/R = \{x_1, \cdots, x_n \mid R_1, \cdots, R_m\}$$

where F is the free group on $x_1, \cdots, x_n$, $n = d(G)$, and R is the normal closure in F of $R_1, \cdots, R_m$, we have always

$$m \geqq r(G) = d(R/[F,R]R^p).$$

We say that G belongs to a class $\mathcal{G}_p$ of finite p-groups if there is a presentation with $n = d(G)$ and $m = r(G)$. Such a presentation is said to be minimal. It is well known (see for example [1]) that if G and H are finite p-groups then

$$r(G \times H) = r(G) + r(H) + d(G)d(H)$$

and hence $G, H \in \mathcal{G}_p$ implies $G \times H \in \mathcal{G}_p$.

The question is asked in [1] whether G, $G \times H \in \mathcal{G}_p$ implies $H \in \mathcal{G}_p$. In this paper we show that if H is any finite p-group, then there exists a finite p-group

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$G \in \mathscr{G}_p$ such that $G \times H \in \mathscr{G}_p$. Hence it follows that G, $G \times H \in \mathscr{G}_p$ implies $H \in \mathscr{G}_p$ if and only if every finite p-group belongs to $\mathscr{G}_p$.

2.

It is shown in [2] that if G is a finite p-group, then $G = F/R$ where F is free on $d(G)$ generators and R is generated as a normal subgroup of F by $r(G)$ elements modulo $[R,R]$. That is if $d(G) = n$, $r(G) = m$ and R has free generators $S_1, \cdots, S_t$, then G has a presentation

$$G = F/R = \{x_1, \cdots, x_n \mid R_1, \cdots, R_m,\ [S_i, S_j];\ i,j = 1,2,\cdots,t\},$$

where $[u,v]$ denotes the commutator $u^{-1}v^{-1}uv$.

Note that t is finite. In fact, $t = 1 + (n-1)|G|$.

Let H be any group in $\mathscr{G}_p$ (say, an elementary abelian group) with $r(H) \geqq t$, presented as

$$H = \{y_1, \cdots, y_{d(H)} \mid T_1, \cdots, T_{r(H)}\}$$

and put

$$\begin{aligned} K \;=\; \{x_1, \cdots, x_n,\ y_1, \cdots, y_{d(H)} \mid\ & R_1, \cdots, R_m, \\ & T_iS_i^{-1}, \quad 1 \leqq i \leqq t, \\ & T_i, \quad t < i \leqq r(H), \\ & [x_i, y_j], \quad 1 \leqq i \leqq n,\ 1 \leqq j \leqq d(H)\} \end{aligned}$$

Then $K = G \times H$ can be seen by observing that the subgroups X, Y of K generated by the x_i, y_j, respectively, centralize each other, and the S_i lie in their intersection, hence in the centre of X. Therefore, $[S_i, S_j] = 1$, and because of the R_i, we have $S_i = 1$ and, therefore, $T_i = 1$.

Therefore the presentation given for K is minimal and K belongs to $\mathscr{G}_p$. However H belongs to $\mathscr{G}_p$ and we have proved the following:

THEOREM. *Let G be a finite p-group. Then G is the direct factor of a member of $\mathscr{G}_p$ by a member of $\mathscr{G}_p$.*

REFERENCES

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